TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 281, Number 1, January 1984

ADDITIVITY OF MEASURE IMPLIES ADDITIVITY OF CATEGORY

BY

TOMEK BARTOSZYŃSKI

ABSTRACT. In this paper it is proved that 2^ω-additivity of category follows from 2^ω-additivity of measure, and a combinatorial characterization of additivity of measure is found.

DEFINITIONS. For abbreviation we denote by $A(c)$, $B(c)$, $A(m)$, $B(m)$, D and hD the following sentences:

$A(c) \equiv$ union of less than 2^ω meager sets is meager.

$B(c) \equiv$ union of less than 2^ω meager sets is not ω^ω.

$A(m)$ and $B(m)$ are defined analogously by replacing the word "meager" by "Lebesgue measure zero". The symbols $\forall^\infty$, $\exists^\infty$ abbreviate "for all but finitely many" and "exist infinitely many". Let $\prec$ denote the following order on ω^ω. For $f, g \in \omega^\omega, f \prec g \equiv \forall^\infty n\, f(n) < g(n)$.

$D \equiv \forall F \subset \omega^\omega [|F| < 2^\omega \to \exists g \in \omega^\omega\, \forall f \in F\, f \prec g]$.

$hD \equiv$ for every family of power less than 2^ω which consists of converging series there exists a converging series eventually dominating each of them.

Recall here some known facts which will be used later.

THEOREM 1 (MILLER, TRUSS). $A(c) \equiv B(c) \,\&\, D$. $\square$

THEOREM 2 (MILLER). *Assuming* D

$$B(c) \equiv \forall F \subset \omega^\omega\left[|F| < 2^\omega \to \exists g \in \omega^\omega\, \forall f \in F\, \exists^\infty n\, f(n) = g(n)\right]. \quad \square$$

THEOREM 3 (KUNEN). $A(c) \nrightarrow A(m)$. $\square$

For the proofs see [**1**].

THEOREM 4 (MILLER). $A(m) \to D$. $\square$

COROLLARY (MILLER). $A(m) \,\&\, B(c) \to A(c)$. $\square$

For the proof of Theorem 4 see [**2**].

Now we will find a combinatorial characterization of $A(m)$ which will be useful to establish the main result.

THEOREM 5. $A(m) \equiv hD$.

Received by the editors December 26, 1982.

1980 *Mathematics Subject Classification*. Primary 03E05; Secondary 28A99.

PROOF. ← We will work in the space R with Lebesgue measure m. Let $\{I_n: n < \omega\}$ be the enumeration of the standard basis of R, i.e. open intervals with rational endpoints. Take any family $\{G_\xi: \xi < \lambda < 2^\omega\}$ of null sets in R. For every $\xi < \lambda$ we can find basic intervals $\{I_n^\xi: n < \omega\}$ such that

$$\forall_{\xi<\lambda} G_\xi \subset \bigcap_n \bigcup_{m>n} I_m^\xi \quad \text{and} \quad \forall_{\xi<\lambda} \sum_{n=1}^{\infty} m(I_n^\xi) < \infty.$$

Now for $\xi < \lambda$ define $f_\xi \in 2^\omega$ as

$$f_\xi(n) = \begin{cases} 1 & I_n = I_k^\xi \text{ for some } k \in \omega \\ 0 & \text{in the other cases} \end{cases} \qquad \text{for } n < \omega.$$

By this definition we have

$$\forall_{\xi<\lambda} \sum_{n=1}^{\infty} f_\xi(n) \cdot m(I_n) < \infty.$$

Now take the family $\{\Sigma_{n=1} f_\xi(n) \cdot m(I_n): \xi < \lambda\}$. It is easy to see that using hD we can get $f \in 2^\omega$ such that $\Sigma_{n=1}^{\infty} f(n) \cdot m(I_n) < \infty$ and $\forall_{\xi<\lambda} \forall^\infty n\ f_\xi(n) \leqslant f(n)$. Let $G = \bigcap_n \bigcup_{m>n, f(m)=1} I_m$. We have

$$G_\xi \subset \bigcap_n \bigcup_{m>n} I_n^\xi \subset G \quad \text{for } \xi < \lambda.$$

This finishes the proof because G is a null set. □

→ LEMMA. *The following conditions are equivalent*:

(i) hD.

(ii) $\forall F \subset \omega^\omega [|F| < 2^\omega \to \exists I_n \subset \omega [|I_n| < n^2 \wedge \forall f \in F \forall^\infty n\, f(n) \in I_n]]$.

(iii)

$$D \,\&\, \forall F \subset \omega^\omega \left[\left(|F| < 2^\omega \,\&\, F \prec f \,\&\, \sum_{n=1}^{\infty} \frac{1}{f(n)} < \infty \right) \right.$$
$$\left. \to \exists I_n \subset \omega \sum_{n=1}^{\infty} \frac{|I_n|}{f(n)} < \infty\ \forall g \in F \forall^\infty n\ g(n) \in I_n \right].$$

PROOF OF THE LEMMA. (i) → (ii). Let $F = \{f_\xi: \xi < \lambda < 2^\omega\} \subset \omega^\omega$. Define for $\xi < \lambda$ the series

$$a_n^\xi = \begin{cases} \max\{k^{-2}: n = f_\xi(k)\}, & n \in Rg(f_\xi), \\ 0, & n \notin Rg(f_\xi). \end{cases}$$

By hD there exists $\Sigma_{n=1}^{\infty} a_n$ eventually dominating all $\Sigma_{n=1}^{\infty} a_n^\xi$, $\Sigma_{n=1}^{\infty} a_n < 1$. Denote $I_k = \{n: a_n \geqslant k^{-2}\}$ for $k < \omega$. Notice that $|I_n| < n^2$ for almost every $n < \omega$ and $\forall g \in F \forall^\infty n\ g(n) \in I_n$. Q.E.D.

(ii) → (i). Take any family of coverging series, i.e. $F = \{f_\xi: \xi < \lambda < 2^\omega\}$ where $f_\xi: \omega \to Q$ (rationals) for every $\xi < \lambda$. Define for $\xi < \lambda$ sequences $\{n_k^\xi: k < \omega\} \subset \omega$

such that

$$\forall_{\xi<\lambda}\forall^{\infty} k \sum_{i>n_k^{\xi}}^{\infty} f_{\xi}(i) < 2^{-k}.$$

Notice that (ii) obviously implies D, so we can find $h \in \omega^{\omega}$ such that

$$\forall_{\xi<\lambda}\forall^{\infty}k\; n_k^{\xi} < h(k).$$

Now define

$$f'_{\xi}(k) = f_{\xi \upharpoonright [h(k), h(k+1))} \quad \text{for } \xi < \lambda.$$

We apply (ii) to the family $\{f'_{\xi}: \xi < \lambda\}$ to get $\{I_n: n < \omega\}$ such that $\forall k\, |I_k| < k^2$ and $\forall_{\xi<\lambda}\forall^{\infty}k\, f'_{\xi}(k) \in I_k$. Each I_k consists of functions from $[h(k), h(k+1))$ to Q. Now define a series $f\colon \omega \to Q$ in the following way:

$$f(n) = \sup\left\{ s(n)\colon s \in I_k \text{ and } \sum_{i=h(k)}^{h(k+1)} s(i) < 2^{-k} \right\}.$$

for $n \in [h(k), h(k+1))$. Notice that $\sum_{n=1}^{\infty} f(n) \leqslant \sum_{n=1}^{\infty} n^2/2^n < \infty$ and $\forall_{\xi<\lambda}\forall^{\infty}n$ $f_{\xi}(n) \leqslant f(n)$, so f is what we were looking for. □

(ii) → (iii). Take any family $F \subset \omega^{\omega}$ such that $|F| < 2^{\omega}$, $F \prec f$ and $\sum_{n=1}^{\infty} 1/f(n) < \infty$. We have to find $\{I_n: n < \omega\}$ such that

$$\sum_{n=1}^{\infty} \frac{|I_n|}{f(n)} < \infty \quad \text{and} \quad \forall g \in F\, \forall^{\infty} n\; g(n) \in I_n.$$

Let $\{R_n: n < \omega\}$ be a nondecreasing sequence such that

$$\sum_{n=1}^{\infty} \frac{R_n}{f(n)} < \infty \quad \text{and} \quad R_n \underset{n\to\infty}{\to} \infty.$$

Pick a sequence $\{u_n: n < \omega\}$ such that $\forall n\; R_{u_n} \geqslant n^2$. For $g \in F$ let g' be defined in the following way:

$$g'(k) = g_{\upharpoonright [u_k, u_{k+1})} \quad \text{for } k < \omega.$$

Applying (ii) to the family $\{g'\colon g \in F\}$ and define as in the previous proof:

$$J_n = \{s(n)\colon s \in I_k\} \quad \text{for } n \in [u_k, u_{k+1}).$$

It is easy to see that

$$\forall g \in F\, \forall^{\infty} n \quad g(n) \in J_n \quad \text{and} \quad \forall^{\infty} n \quad |J_n| \leqslant R_n.$$

As we noticed earlier (ii) → D so the implication (ii) → (iii) is proved. □

(iii) → (ii). Take any $F \subset \omega^{\omega}$, $|F| < 2^{\omega}$. By D we can find $f \in \omega^{\omega}$ and a strictly increasing sequence $\{k_n: n < \omega\} \subset \omega$ such that $F \prec f$ and $k_n/f(n) = n^{-2}$ for every $n < \omega$. For $g \in F$ define $g' \in \omega^{\omega}$ in the following way:

$$g'\colon \Big|\underbrace{\;g(1)\;}_{k_1 \text{ times}}\Big|\underbrace{\;g(2)\;}_{k_2 \text{ times}}\Big|\underbrace{\;g(3)\;}_{k_3 \text{ times}}\Big|$$

Now apply (iii) to the family $\{g'\colon g \in F\}$ and define $J_n = I_j$ of minimal power from the k_nth block for $n < \omega$. We have

$$\infty > \sum_{n=1}^{\infty} \frac{|I_n|}{f'(n)} \geqslant \sum_{n=1}^{\infty} \frac{k_n|J_n|}{f(n)} = \sum_{n=1}^{\infty} \frac{|J_n|}{n^2}.$$

Thus $|J_n| < n^2$ for almost every $n < \omega$ and $\forall g \in F\, \forall^\infty n\; g(n) \in J_n$. This finishes the proof of the lemma. □

Now we will prove that $A(m) \to$ (iii). By Theorem 4 we already know that $A(m) \to D$. Take any family $F \subset \omega^\omega$, $F \prec f$ such that $|F| < 2^\omega$ and $\Sigma_{n=1} f(n)^{-1} < \infty$. Let $X = \prod_{n=1}^{\infty} f(n)$ and

$$H_g = \{x \in X\colon \exists^\infty n\; x(n) = g(n)\} \quad \text{for } g \in X.$$

Denote by μ the standard product measure on X.

$$\mu(H_g) = \mu\Big(\bigcap_n \bigcup_{m>n} \{x \in X\colon x(m) = g(m)\}\Big)$$

$$\leqslant \mu\Big(\bigcup_{m>n} \{x \in X\colon x(m) = g(m)\}\Big) \leqslant \sum_{m>n} f(m)^{-1} \to 0 \quad \text{as } n \to \infty.$$

Thus $\mu(H_g) = 0$ for $g \in X$. By $A(m)$ we can find a closed set, i.e. a tree T such that the set of its branches $[T]$ has positive measure and $[T] \cap H_g = \varnothing$ for $g \in F$. Denote $T(n) = \{x(n)\colon x \in [T]\}$ and $T_s = \{t \in T\colon s \leqslant t\}$. We can assume that $[T_s]$ has positive measure for $s \in T$.

Claim. $\forall g \in F\, \exists s \in T\, \forall n > lh(s)\; g(n) \notin T_s(n)$.

Proof. Assume that it is false for some $g \in F$. Then there is a branch $x \in [T]$ such that $\exists^\infty n\; x(n) = g(n)$. But this means that $[T] \cap H_g \neq \varnothing$, thus a contradiction. □

Therefore for every $g \in F$ we have some subtree T_s as in the claim: let us call them $T_{s_1}, T_{s_2}, T_{s_3}, \ldots$. Denote $I_m^n = T_{s_n}(m)$ for $n, m < \omega$. It is easy to see that

$$\prod_{m=1}^{\infty} \frac{|I_m^n|}{f(m)} > 0 \quad \text{for all } n < \omega.$$

By changing the first few I_m^n's for every $n < \omega$ we can obtain $\{I_m^n\colon n, m < \omega\}$ (use the same name) such that

$$\prod_{m=1}^{\infty} \frac{|I_m^n|}{f(m)} > 1 - 2^{-n-1} \quad \text{for } n < \omega.$$

Now let $J_m = \bigcap_n I_m^n$ for $m < \omega$. Obviously $\prod_n |J_n|/f(n) > 0$. Let $I_n = f(n) - J_n$, $n < \omega$. It is not very hard to see that $\Sigma_{n=1}^{\infty} |I_n|/f(n) < \infty$ and $\forall g \in F \forall^\infty n\; g(n) \in I_n$. This finishes the proof of Theorem 5. □

Theorem 6. $hD \to A(c)$.

Corollary. $A(m) \to A(c)$. □

Proof. By Theorems 1 and 2 in order to prove Theorem 6 it is enough to show that (ii) implies

$$\forall F < \omega^\omega\big[|F| < 2^\omega \to \exists g \in \omega^\omega\, \forall f \in F\, \exists^\infty n\; f(n) = g(n)\big].$$

Take any family $F \subset \omega^\omega$, $|F| < 2^\omega$. We have to find $g \in \omega^\omega$ such that $\forall f \in F \exists^\infty n$ $f(n) = g(n)$. For $f \in F$ define f' in the following way:

$$f'(k) = f_{\restriction [k^3, (k+1)^3)}, \qquad k < \omega.$$

Applying (ii) to the family $\{f' : f \in F\}$ we obtain $\{I_n : n < \omega\}$ such that $|I_n| < n^2$ for $n < \omega$ and $\forall f \in F \forall^\infty n$ $f'(n) \in I_n$. Notice that each I_n consists of less than n^2 functions whose domains have power bigger than n^2 (because $(n+1)^3 - n^3 > n^2$). Thus it is easy to find the required function picking one value out of each function from I_n for every $n < \omega$. This finishes the proof of Theorem 6. □

In fact we have proved

COROLLARY. *If unions of less than λ null sets are null than unions of less than λ meager sets are meager.* □

This was also proved later by J. Raisonnier and Y. Stern.

REFERENCES

1. Arnold Miller, *Some properties of measure and category*, Trans. Amer. Math. Soc. **266** (1981), 93–114.
2. ______, *Additivity of measure implies dominating real*, Proc. Amer. Math. Soc. (to appear).

00-785 WARSAW, GROTTGERA 12A-51, POLAND